Flow and Heat Transfer on a Disk Rotating Beneath a Forced Vortex

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The integral method of von Kármán for calculating the turbulent boundary layer on a rotating disk has been extended to cases for which the outer flow is rotating at a constant angular velocity. The method predicts the nondimensional radial mass flux, the windage, and, by analogy, the Nusselt number. It enables quick estimates to be made for the flow over the rear of compressor and turbine disks.

Nomenclature

$C(n)$	= constant in the power law of the law of the wall
C_1-C_6	= constants defined in terms of n
C_p	= specific heat at constant pressure
G	= constant of proportionality in the heat-transfer analogy
H	= stagnation enthalpy
K	= rotary core velocity divided by ωr
$\dot{m}$	= radial mass flux in the boundary layer
m	$=(1-n)/(1+3n)$
M	= turning moment on the disk
n	= exponent in the boundary-layer velocity profile
Nu	= Nusselt number
Pr	= Prandtl number $=\mu C_p/\lambda$
Q_w	= windage $=M\omega$
r	= radius
R_θ	= Reynolds number $=\theta\omega r/\nu$
Re	= Reynolds number $=r\omega(1-K)\delta/\nu$
R_r	= Reynolds number $=\omega r^2/\nu$
s	$=[(r\omega-v)^2+u^2]^{1/2}$
S	$=[[r\omega(1-K)]^2+[r\alpha\omega]^2]^{1/2}$
T_{aw}	= adiabatic wall temperature
T_w	= wall temperature
T_0	= stagnation temperature of the surrounding core flow
u	= radial velocity
v	= tangential velocity relative to room coordinates
z	= distance from the disk
α	= parameter determining the magnitude of the radial flow
β	= parameter in the equation for boundary-layer thickness
δ	= boundary-layer thickness
θ	= tangential momentum thickness
λ	= thermal conductivity
λ_e	= eddy thermal conductivity in turbulent flow
μ	= viscosity
μ_e	= eddy viscosity in turbulent flow
ν	= kinematic viscosity
ρ	= density
τ_w	= skin friction at the wall
τ_r	= radial component of τ_w
τ_θ	= tangential component of τ_w
ω	= angular velocity of the disk

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Introduction

THE temperature distribution in the impeller disk of high-speed centrifugal compressors must be determined in order to develop designs which will not fail structurally. Typically there is a transfer of heat into the disk from the air within the rear cavity and the highest temperatures are encountered at midradii. In practice, the temperature may be reduced by introducing cooling air either through a radial seal at the tip of the disk or directly into the cavity. This additional air has two effects. First, it directly reduces the temperature within the core flow of the cavity, and second, it may lead to an overall rotation of the flow in the core in the same direction as that of the disk, which has the effect of reducing the heat transfer to the disk. Recently, approximate methods have been developed for numerically analyzing the flow in the seal and the cavity,[1] and these predictions have been compared with a new set of test data. Both the predictions and the test data indicate that in many situations the flow in the core of the cavity is rotating over most of cavity as a forced vortex, i.e., the core flow has a rotational velocity V which is proportional to the radius r. V is usually in the same direction as the motion of the disk, and is often written as $K\omega r$, where ω is the angular velocity of the disk. Thus K is approximately constant for the core flow.

In his original integral analysis for the incompressible boundary-layer flow over a rotating disk in still surroundings ($K=0$), von Kármán[2] showed that the radial velocity profile was similar at all radii and that the boundary-layer thickness was proportional to the radius raised to a fixed exponent. The differential equations for the tangential and radial flow became simple algebraic equations and a solution was obtained without recourse to numerical techniques. It turns out that the same is true of *all* incompressible cases with K constant but not necessarily zero. The purpose of the present report is to extend von Kármán's analysis for these cases. Nondimensional forms are developed for the boundary-layer thickness, the radial mass flux in the boundary layer, and the energy required to rotate the disk (windage). In addition, the Nusselt number is calculated on the basis of the Dorfman-Owen analogy between the heat flux and the tangential skin friction. The results are presented in graphical form as a function of K and enable quick estimates to be made for the above quantities.

Analysis

Consider one side of the disk rotating with angular velocity ω and surrounded by a fluid which is also rotating with constant angular velocity $K\omega$ (Fig. 1). At radius r and distance z from the disk the velocity components of the fluid are radial outwards (u) and tangential in the same direction as the motion of the disk (v). Thus at $z=0$, $u=0$, and $v=\omega r$ and for $z=\delta$, the boundary-layer thickness, $u=0$, $v=K\omega r$.

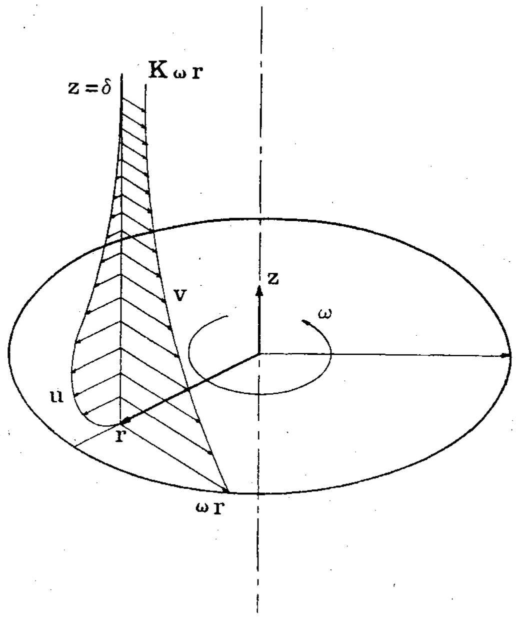

Fig. 1 Velocity distribution near the disk.

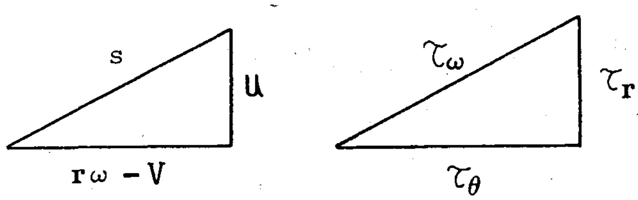

Fig. 2 Velocity relative to the disk and the skin friction.

Following von Kármán,[2] power-law velocity profiles are assumed which conform to these boundary conditions:

$$u = (z/\delta)^n r\alpha\omega[1-(z/\delta)]$$

$$v - K\omega r = r\omega(1-K)[1-(z/\delta)^n] \tag{1}$$

where α is a number which usually varies with r and determines the magnitude of the radial flow.

The skin friction components in the radial and tangential directions, τ_r and τ_θ, respectively, are each assumed to be proportional to the gradient of the velocity in the appropriate direction near the wall. Thus

$$\frac{\tau_r}{\tau_\theta} = \lim_{z\to 0}\left(\frac{u}{r\omega - v}\right) = \frac{r\alpha\omega}{r\omega(1-K)} \tag{2}$$

Moreover the total skin friction τ_w determines the variation of the total velocity s near the wall (Fig. 2).

Near the wall,

$$\frac{r\omega - v}{r\omega(1-K)} = \left(\frac{z}{\delta}\right)^n = \frac{u}{r\alpha\omega}$$

$$= \frac{s}{\{[r\omega(1-K)]^2 + [r\alpha\omega]^2\}^{1/2}} = \frac{s}{S} \tag{3}$$

and to conform to the law of the wall,

$$\frac{s}{(\tau_w/\rho)^{1/2}} = C(n)\left[\frac{z(\tau_w/\rho)^{1/2}}{\nu}\right]^n \tag{4}$$

where $C(n)$ is a constant which depends on the chosen exponent n.

Table 1 Exponent n and associated constant $C(n)$ as a function of Re

n	1/7	1/8	1/9	1/10
$C(n)$	8.74	9.71	10.6	11.5
$Re\sim$	$<0.6\times10^5$	3×10^5	7×10^5	17×10^5

Both $C(n)$ and n are chosen so that the power law conforms to the more general logarithmic law of the wall, and they therefore depend on the Reynolds number of the flow $r\omega(1-K)\delta/\nu$. The values given by Schlichting[3] are shown in Table 1.

From Eqs. (3) and (4),

$$\frac{\tau_w}{\rho S^2} = \left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\left[\frac{\nu}{\delta S}\right]^{\frac{2n}{n+1}} \tag{5}$$

where $S^2 = [r\omega(1-K)]^2 + [r\alpha\omega]^2$.

From Eqs. (2) and (3),

$$\frac{\tau_r}{\tau_w} = \frac{r\alpha\omega}{S} \qquad \frac{\tau_\theta}{\tau_w} = \frac{r\omega(1-K)}{S} \tag{6}$$

The integral momentum equation for the incompressible boundary layer in the tangential direction is

$$-K\omega r^2\frac{\mathrm{d}}{\mathrm{d}r}\left[r\int_0^\delta u\mathrm{d}z\right] + \frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\int_0^\delta uv\mathrm{d}z\right] = \frac{\tau_\theta}{\rho}r^2$$

The integral momentum equation in the radial direction is

$$-\frac{1}{r}\int_0^\delta[v^2 - (K\omega r)^2]\mathrm{d}z + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\int_0^\delta u^2\mathrm{d}z\right] = \frac{-\tau_r}{\rho}$$

Substituting Eqs. (1), (5), and (6), these equations become tangential:

$$-K\omega r^2\frac{\mathrm{d}}{\mathrm{d}r}[r^2\delta\alpha\omega C_1] + \frac{\mathrm{d}}{\mathrm{d}r}\{r^4\delta\omega^2[\alpha(C_1 - C_2) + KC_2]\}$$
$$= r^3\omega(1-K)\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\left(\frac{\nu}{\delta}\right)^{\frac{2n}{n+1}}$$
$$\times\left\{[r\omega(1-K)]^2 + [r\alpha\omega]^2\right\}^{\frac{1-n}{2(n+1)}} \tag{7}$$

and radial:

$$- -\delta r^2\omega^2[C_4 + KC_5 + K^2C_6] + \frac{\mathrm{d}}{\mathrm{d}r}[r^3\delta(\alpha\omega)^2C_3] = -r^2\alpha\omega$$
$$\times\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\left(\frac{\nu}{\delta}\right)^{\frac{2n}{n+1}}\left\{[r\omega(1-K)]^2 + [r\alpha\omega]^2\right\}^{\frac{1-n}{2(n+1)}} \tag{8}$$

where

$$C_1 = \frac{1}{n+1} - \frac{1}{n+2} \qquad C_2 = \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$C_3 = \frac{1}{2n+1} - \frac{2}{2n+2} + \frac{1}{2n+3} \qquad C_4 = 1 - \frac{2}{n+1} + \frac{1}{2n+1}$$

$$C_5 = 2\left[\frac{1}{n+1} - \frac{1}{2n+1}\right] \qquad C_6 = \frac{1}{2n+1} - 1$$

In general, the boundary-layer thickness δ, the weighting factor for the radial velocity α and K all vary with r in the above equations. However, if K is constant, a similarity solution with α also constant is possible if $\delta=\beta r^m$, where β is a constant. The dependence on r is then eliminated from the differential equations (7) and (8) if $m=(1-n)/(1+3n)$ and they become simultaneous equations for α and β, as first demonstrated by von Kármán[2] for the case $K=0$.

The equations are

$$-K(m+2)\alpha C_1\beta+(m+4)\beta\alpha(C_1-C_2+KC_2) = \left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}(1-K)\left(\frac{\nu}{\beta\omega}\right)^{\frac{2n}{n+1}}\left\{\alpha^2+(1-K)^2\right\}^{\frac{1-n}{2(n+1)}} \tag{9}$$

$$-\beta(C_4+KC_5+K^2C_6)+(m+3)\alpha^2C_3\beta = -\alpha\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\left(\frac{\nu}{\beta\omega}\right)^{\frac{2n}{n+1}}\left\{\alpha^2+(1-K)^2\right\}^{\frac{1-n}{2(n+1)}} \tag{10}$$

Dividing Eqs. (9) and (10), β is eliminated,

$$\alpha^2=[(C_4+KC_5+K^2C_6)(1-K)]/[(1-K)(m+3)C_3 + (m+4)(C_1-C_2+KC_2)-K(m+2)C_1] \tag{11}$$

In the above equation, m and $C(n)$ are known functions of the chosen exponent n. Thus α is a known function of K and is plotted in Fig. 3.

Substituting back into Eq. (10) the β Reynolds number,

$$\beta\left(\frac{\omega}{\nu}\right)^{\frac{2n}{3n+1}} = \left\{\frac{\alpha\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\left[\alpha^2+(1-K)^2\right]^{\frac{1-n}{2(n+1)}}}{C_4+KC_5+K^2C_6-(m+3)\alpha^2C_3}\right\}^{\frac{n+1}{3n+1}} \tag{12}$$

and is plotted in Fig. 4.

The associated prediction of boundary-layer thickness is similar to the theoretical curve expressed in terms of Y' in Fig. 41 of Ref. 7. Quantities of specific interest may now be calculated as a function of K.

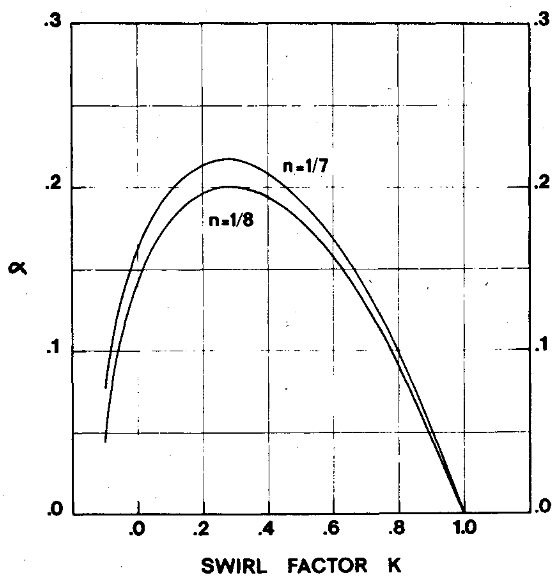

Fig. 3 Radial flow factor α as a function of K.

Radial Mass Flux in the Boundary Layer at Radius r

For the boundary layer which has grown from $r=0$, the radial mass flux

$$\dot{m}=\rho\int_0^{\delta}2\pi r u dz=\rho 2\pi C_1\omega\alpha\beta r^{m+2}$$

This mass flux may be expressed nondimensionally as

$$\frac{\dot{m}}{\pi\rho r^3\omega}=2\alpha\beta\left(\frac{\omega}{\nu}\right)^{\frac{2n}{3n+1}}C_1R_r^{-\frac{2n}{3n+1}} \tag{13}$$

Hence $(\dot{m}/\pi\rho r^3\omega)(R_r)^{2n/(3n+1)}$ is a unique function of K and is shown in Fig. 5.

Windage to Radius r

The power required to turn one side of a disk of radius r is also the mechanical energy supplied to the flow in unit time and appears in the overall energy balance for the cavity as the windage.[1] Thus the windage $Q_w=M\omega$, where M is the turning moment.

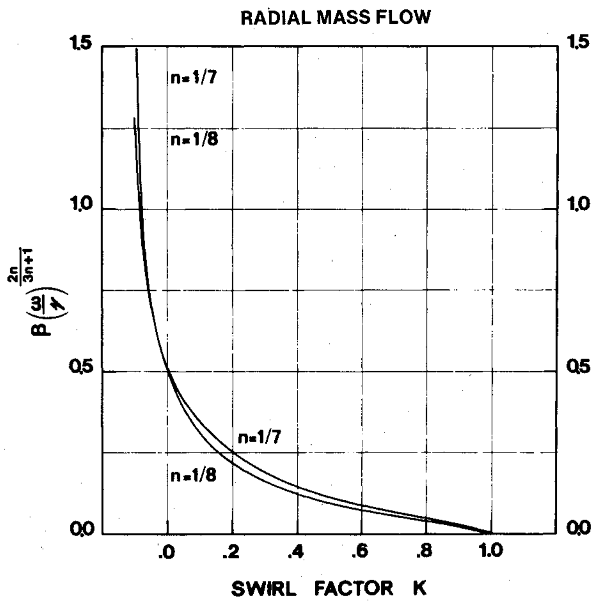

Fig. 4 Boundary-layer thickness as a function of K.

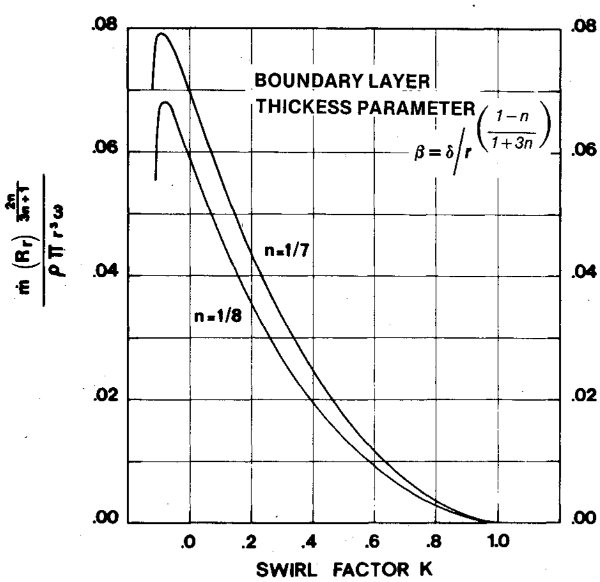

Fig. 5 Boundary-layer mass flux as a function of K.

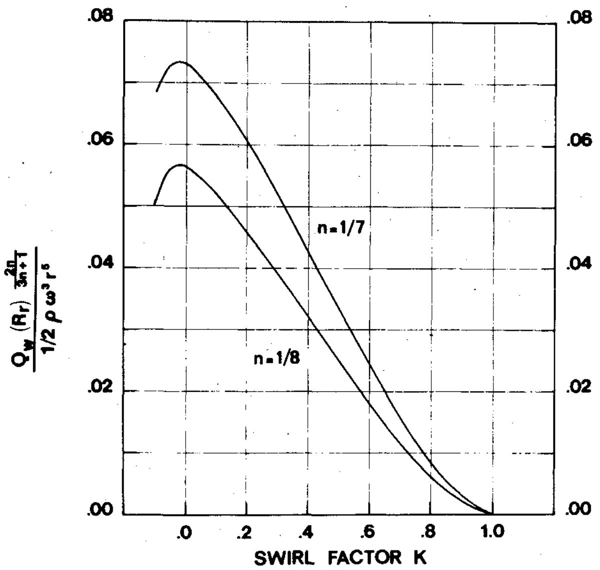

Fig. 6 Windage as a function of K.

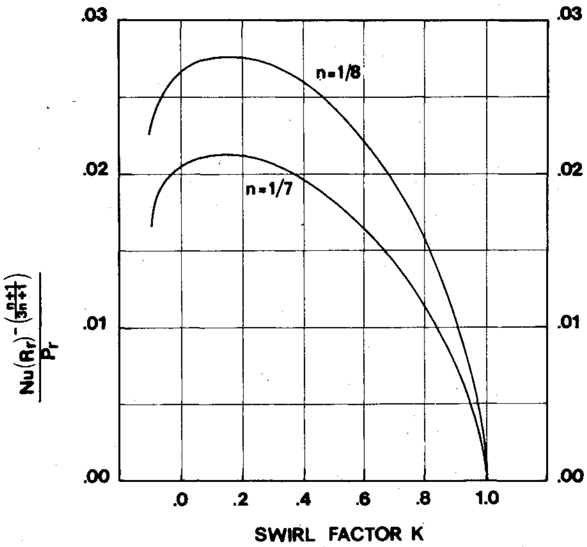

Fig. 7 Nusselt number as a function of K.

$$Q_w=\int_0^r 2\pi r^2\omega\tau_\theta dr$$

$$=2\pi\rho\omega\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\left(\frac{\nu}{\beta}\right)^{\frac{2n}{n+1}}\omega(1-K)$$

$$\times\left\{\omega^2\left[\alpha^2+(1-K)^2\right]\right\}^{\frac{1-n}{2(n+1)}}\frac{r^{m+4}}{m+4}$$

whence

$$\frac{Q_w}{\frac{1}{2}\rho\omega^3r^5}(R_r)^{\frac{2n}{3n+1}}=\frac{2}{m+4}\left\{2\pi\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\right.$$

$$\left.\times(1-K)\left[\alpha^2+(1-K)^2\right]^{\frac{1-n}{2(n+1)}}\right\}\left[\beta\left(\frac{\omega}{\nu}\right)^{\frac{2n}{3n+1}}\right]^{\frac{-2n}{n+1}}\quad(14)$$

Again, this is a unique function of K for each value of n and is shown in Fig. 6.

Nusselt Number

Owen[4] has extended Dorfman's analogy between energy and tangential momentum to compressible flow. The analogy assumes a perfect gas and a turbulent Prandtl number of unity. The stagnation, or total, enthalpy, $H=C_pT+\frac{1}{2}(u^2+v^2)$ is linearly related to the tangential momentum rv,

$$H-H_{\text{ref}}=Grv$$

At the disk, $u=0$, $v=\omega r$, $T=T_w$, and far away, $u=0$, $v=K\omega r$, $H=C_pT_0$, where T_0 is the stagnation temperature of the surrounding core flow at radius r. Hence

$$C_p(T_w-T_0)+\frac{1}{2}\omega^2r^2=Gwr^2(1-K)$$

Thus T_w-T_0 must vary as the square of the radius for the analogy to apply.

The heat flux *into* the wall per unit area of disk is

$$q=\left(\lambda_e\frac{\partial T}{\partial z}\right)_{z=0}=\mu_e\left\{\frac{\partial}{\partial z}\left[H-\frac{1}{2}(u^2+v^2)\right]_{z=0}\right\}$$

$$=\mu_e\frac{\partial}{\partial z}\left[Grv-\frac{1}{2}(u^2+v^2)\right]_{z=0}$$

for turbulent Prandtl number $\mu_eC_p/\lambda_e=1$.

At $z=0$, $u=0$, $v=\omega r$, and $-\mu_e(\partial v/\partial z)_{z=0}=\tau_\theta$.

Thus

$$q=\tau_\theta\left[\frac{C_p(T_0-T_w)+\omega^2r^2(\frac{1}{2}-K)}{\omega r(1-K)}\right]\quad(15)$$

For an adiabatic wall $q=0$ and the wall temperature,

$$T_{aw}=T_0+(\omega^2r^2/C_p)(\frac{1}{2}-K)\quad(16)$$

The Nusselt number, Nu is defined in terms of $(T_{aw}-T_w)$ and the radius r,

$$Nu=\frac{qr}{\lambda(T_{aw}-T_w)}=\frac{\tau_\theta C_p}{\lambda\omega(1-K)}=\frac{\tau_\theta C_pP_r}{\mu\omega(1-K)}$$

where Pr is the laminar Prandtl number.

$$Nu=\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\frac{rPr}{\nu}\left(\frac{\nu}{\beta r^m}\right)^{\frac{2n}{n+1}}r^2\omega^2\left[\alpha^2+(1-K)^2\right]^{\frac{1-n}{2(n+1)}}$$

$$\frac{Nu}{Pr}(R_r)^{-\frac{n+1}{3n+1}}=\left[\frac{1}{C(n)}\right]^{\frac{2}{n+1}}\left[\alpha^2+(1-K)^2\right]^{\frac{1-n}{2(n+1)}}$$

$$\times\left\{\beta\left[\frac{\omega}{\nu}\right]^{\frac{2n}{3n+1}}\right\}^{\frac{-2n}{n+1}}\quad(17)$$

The right-hand side of Eq. (17) is a function of K for a chosen value of n. It is plotted in Fig. 7.

Discussion of Results and Conclusions

The present solutions for various values of K are for a boundary layer which is turbulent over the *whole* of the disk and has grown from zero thickness at the center of the disk. The latter follows from the requirement that for these self-similar solutions, $\delta=\beta r^m$, where β and m are positive constants.

In reality, the boundary layer is laminar near the center of the disk. For the case $K=0$ the boundary layer becomes unstable at $R_r=1.85\times10^5$ and fully turbulent at $R_r=2.85\times10^5$ (Refs. 5, 6). Information on transition for cases with $K\neq0$ is sparse. It might be anticipated that the critical values of R_r would increase with increasing K because the relative angular velocity between the disk and its surroundings is reduced even though the radial pressure gradient is increased. However the experiments in Ref. 7 for an enclosed

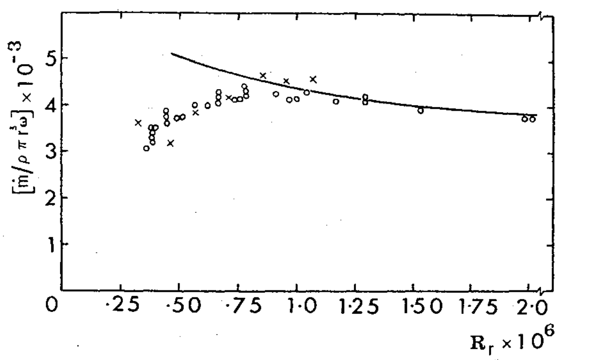

Fig. 8 Boundary-layer mass flux as function of R_r for $K=0$. Theory: (—), $n=1/7$. Experiment: × (Ref. 8), ○ (Ref. 5).

Table 2 R_θ at instability and at transition

R_r	Laminar	Fully turbulent, $n=1/7$
1.85×10^5	647	837
2.85×10^5	803	1182

disk indicate that critical values of R_r are *reduced* as K increases, for example, with $K=0.5$, $R_r=2\times10^5$ at transition. This reduction may be partly due to enhanced turbulence in the core flow when the disk is enclosed. In the absence of further information, therefore, it is reasonable to assume that the transition Reynolds number does not vary appreciably with K.

The appropriate choice for low Reynolds numbers is $n=1/7$; $n=1/8$ is only appropriate when $R_\delta>3\times10^5$, which corresponds to $R_r>160\times10^5$. Comparison with available experiment results for $K=0$ and $R_r<20\times10^5$ is therefore made using von Kármán's theory, with $n=1/7$.

For $K=0$, the von Kármán solution gives an accurate prediction of the tangential momentum thickness θ for $R_r<20\times10^5$ (Fig. 6 in Ref. 5). This may be attributed to the similarity of the laminar, and the calculated turbulent, values of θ in the vicinity of transition. For the laminar boundary layer, θ is constant, independent of the radius r, and is given by[2]

$$\theta(\omega/\nu)^{1/2}=1.505$$

Thus the values of $R_\theta=\theta r\omega/\nu$ at the unstable and the transition Reynolds numbers are shown in Table 2. The values are therefore similar and this probably explains the comparatively good agreement between theory and experiment in Fig. 6 of Ref. 5 for $R_r>5\times10^5$.

In Cham and Head's paper (Fig. 10 of Ref. 5) the skin friction is seen to be well predicted by von Kármán's theory, using $n=1/7$ for $R_r<20\times10^5$. The predicted radial mass flux is compared with experiment in Fig. 8 and is only satisfactory for $R_r>8\times10^5$. For values of R_r between 3×10^5 and 8×10^5, where the entrainment is significantly affected by the initial region of laminar flow, it would be more accurate to apply Head's entrainment method of analysis[5]; but this has not yet been done for non-zero values of K.

When the present theory is used to estimate the windage and Nusselt number on the rear of the centrifugal impeller in a typical engine, the boundary layer starts at a *finite* radius (typically $R_r=5\times10^5$) so that the flow may be taken as turbulent there. The tip Reynolds number is usually about ten times as great as the root value so that the error due to the omission of the central portion of the disk is not serious. The present theory may therefore be used for making preliminary estimates using an assumed value of K typical of that in the core flow of the cavity. The values are usually less than 0.5 for typical cooling flows and typical obstructions in the cavity. It is worth noting that for this range of K the Nusselt number is constant within $\pm6\%$. The change in heat transfer is then mainly due to the change of adiabatic wall temperature as given by Eq. (16).

The present theory enables rapid estimates to be made for the radial mass flux in the rotating boundary layer, the windage, and the Nusselt number for an assumed value for the fluid rotation in the outer flow, the angular velocity of which, $K\omega$, is assumed to be constant. These results have been presented for $K<1.0$. The methods seem to fail for negative values of K (the core flow rotating in the opposite direction to the disk) when K is less than about -0.1. When $K>1.0$, α becomes negative and the radial flow is inwards. Results for these cases are not presented here but could readily be generated and may be of interest for turbine disks in which the core rotation is augmented by jets.

Acknowledgment

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